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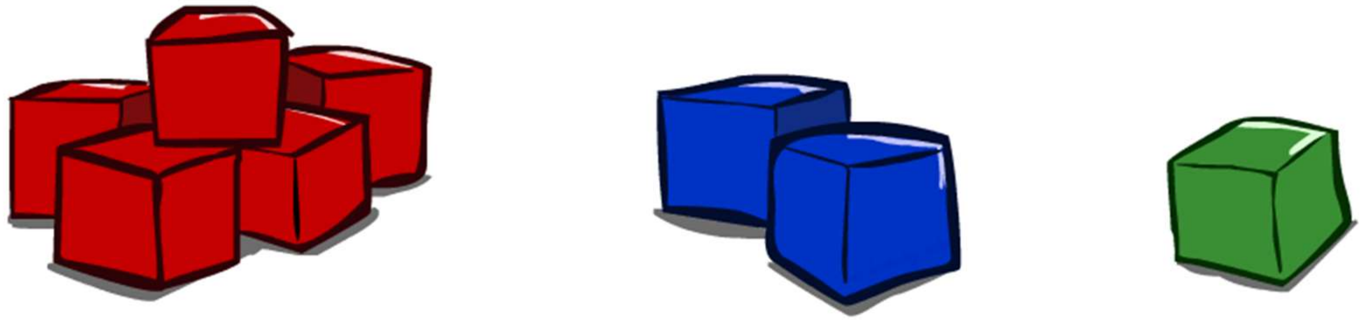
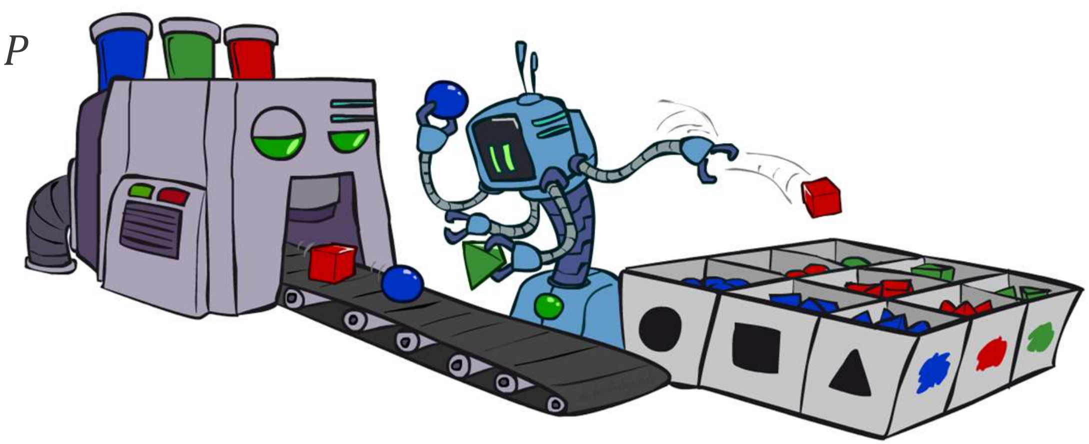


Approximate Inference: Sampling

Sampling

- Sampling is a lot like repeated simulation
  - Predicting the weather, basketball games, …
- Basic idea
  - Draw N samples from a sampling distribution S
  - Compute an approximate posterior probability
  - Show this converges to the true probability P

- Why sample?
  - Learning: get samples from a distribution you don't know
  - Inference: getting a sample is faster than computing the right answer (e.g. with variable elimination)



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Sampling

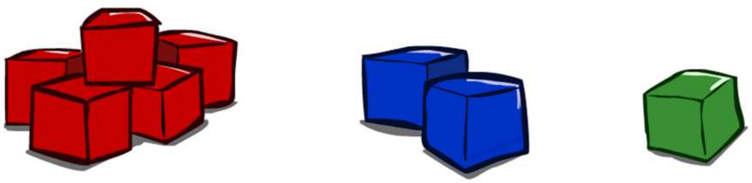
- Sampling from given distribution
  - Step 1: Get sample u from uniform distribution over [0, 1)
    - E.g. random() in python
  - Step 2: Convert this sample u into an outcome for the given distribution by having each outcome associated with a sub-interval of [0,1) with sub-interval size equal to probability of the outcome

- Example

C	$P(C)$
red	0.6
green	0.1
blue	0.3

$$0 \leq u < 0.6, \rightarrow C = red$$
$$0.6 \leq u < 0.7, \rightarrow C = green$$
$$0.7 \leq u < 1, \rightarrow C = blue$$

- If random() returns $u = 0.83$, then our sample is C = blue
- E.g, after sampling 8 times:



Sampling in Bayes' Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

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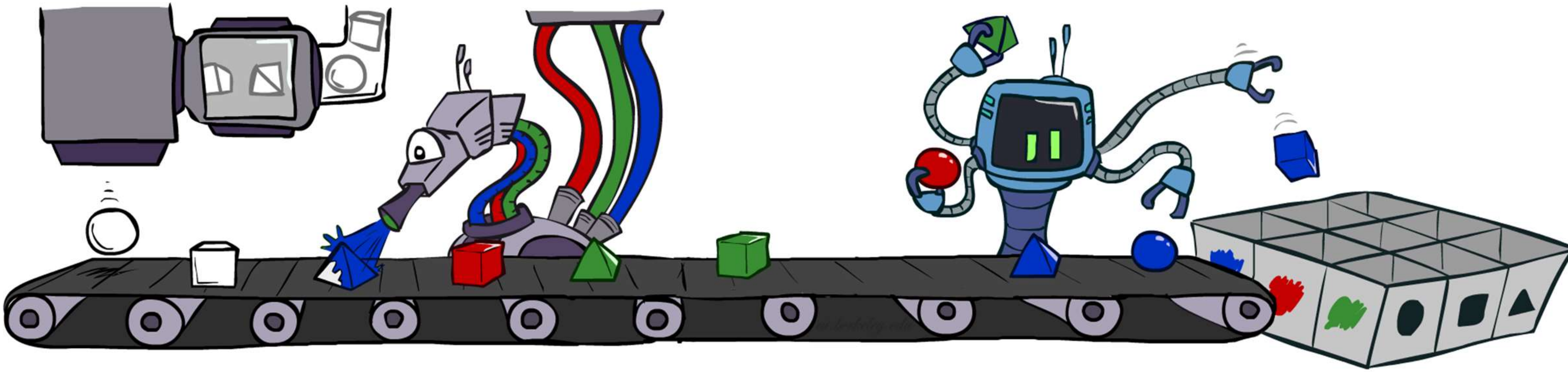
Prior Sampling



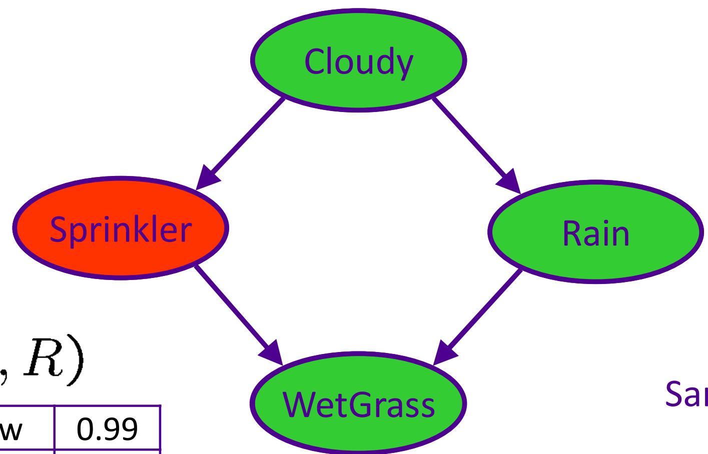
Prior Sampling

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

+c, -s, +r, +w

-c, +s, -r, +w

…

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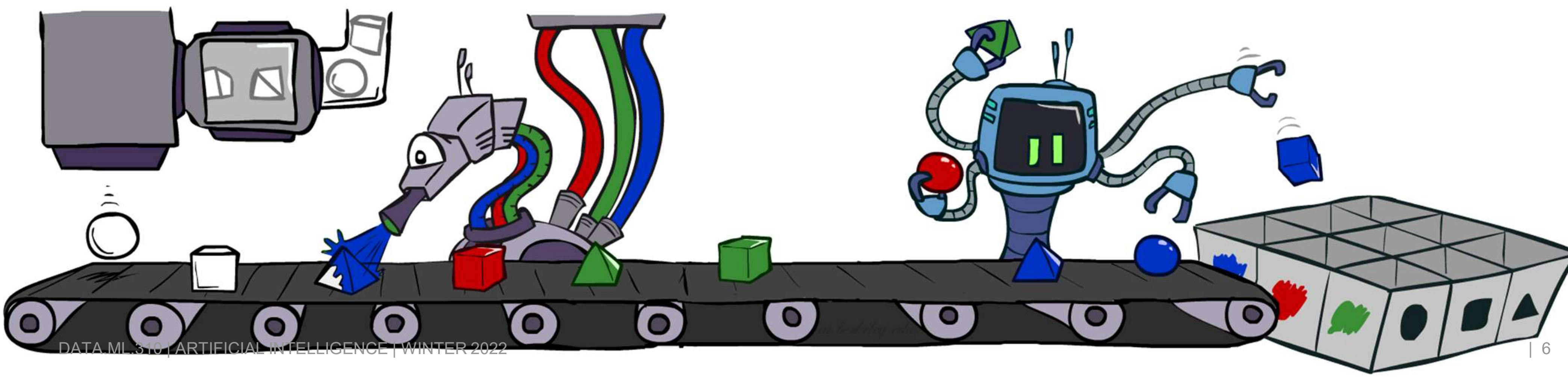
Prior Sampling

for $i = 1, 2, \dots, n$

Sample x_i from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, \dots, x_n)$



Probabilities in BNs

- Why are we guaranteed that setting

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

results in a proper joint distribution?

- Chain rule (valid for all distributions):

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1 \ldots x_{i-1})$$

- Assume conditional independences:

$$P(x_i | x_1, \ldots x_{i-1}) = P(x_i | \mathit{parents}(X_i))$$

→ Consequence:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

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Prior Sampling

- This process generates samples with probability:

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|\text{Parents}(X_i)) = P(x_1 \ldots x_n)$$

…i.e. the BN's joint probability

- Let the number of samples of an event be $N_{PS}(x_1 \ldots x_n)$

- Then $$\begin{aligned} \lim_{N\to\infty} \widehat{P}(x_1,\ldots,x_n) &= \lim_{N\to\infty} N_{PS}(x_1,\ldots,x_n)/N \\ &= S_{PS}(x_1,\ldots,x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

- I.e., the sampling procedure is consistent

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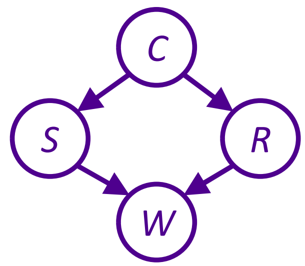
Example

- We' ll get a bunch of samples from the BN:

  $+c, -s, +r, +w$
  $+c, +s, +r, +w$
  $-c, +s, +r, -w$
  $+c, -s, +r, +w$
  $-c, -s, -r, +w$



- If we want to know $P(W)$
  - We have counts $\langle +w: 4, -w: 1 \rangle$
  - Normalize to get $P(W) = \langle +w: 0.8, -w: 0.2 \rangle$
  - This will get closer to the true distribution with more samples
  - Can estimate anything else, too
  - What about $P(C| + w)$? $P(C| + r, +w)$? $P(C| - r, -w)$?
  - Fast: can use fewer samples if less time (what's the drawback?)

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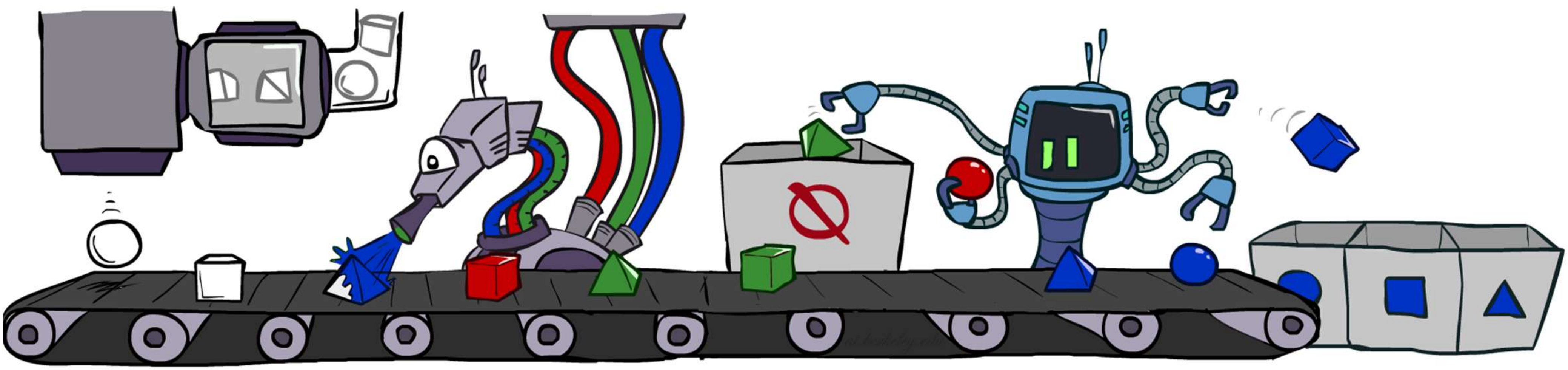
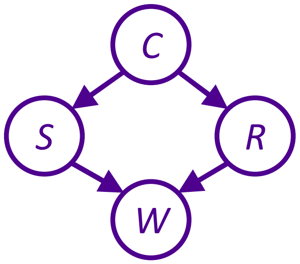
Rejection Sampling



Rejection Sampling

- Let's say we want $P(C)$
  - No point keeping all samples around
  - Just tally counts of C as we go

- Let's say we want $P(C| + s)$
  - Same thing: tally C outcomes, but ignore (reject) samples which don't have $S = +s$
  - This is called rejection sampling
  - It is also consistent for conditional probabilities (i.e., correct in the limit)



+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

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Rejection Sampling

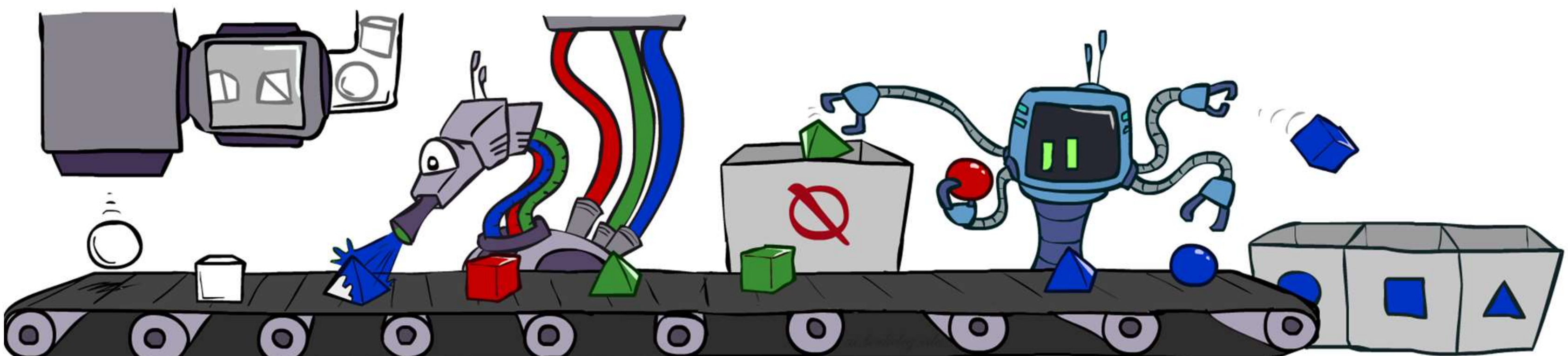
IN: evidence instantiation

for $i = 1, 2, \ldots, n$

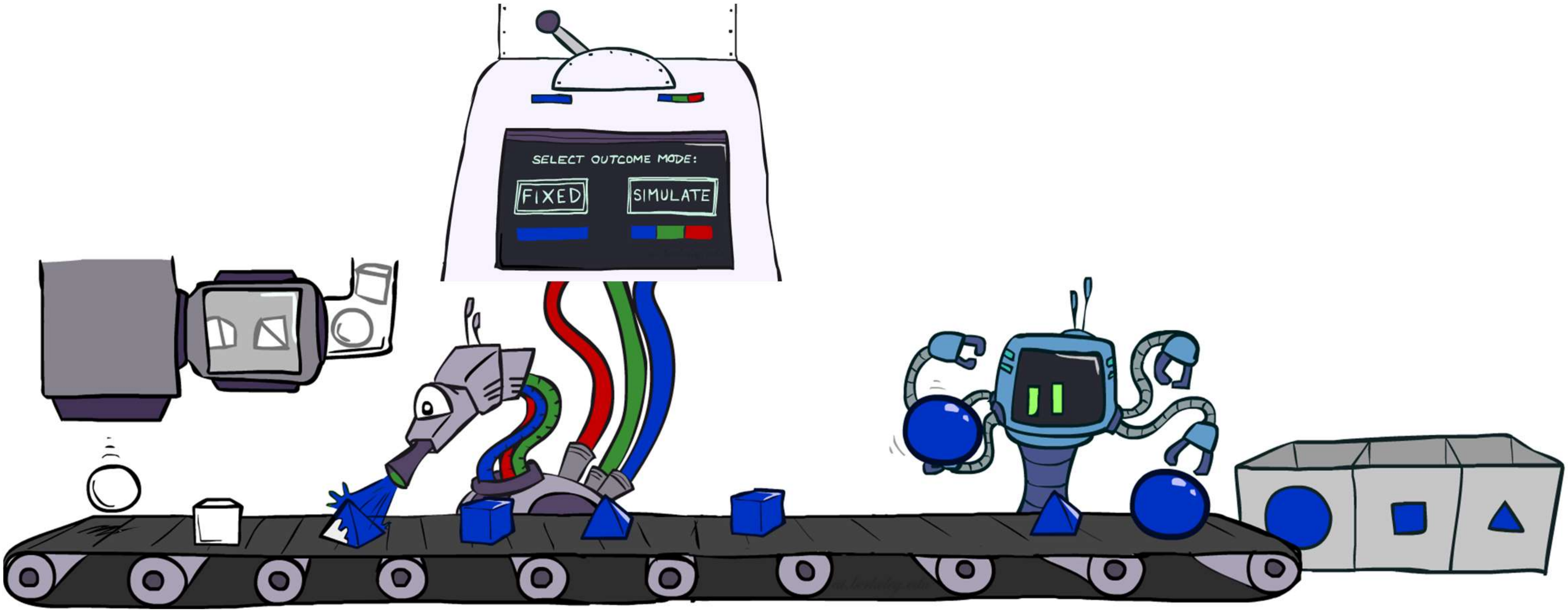
Sample x_i from $P(X_i \mid Parents(X_i))$

if x_i not consistent with evidence

reject: Return, and no sample is generated in this cycle



Likelihood Weighting



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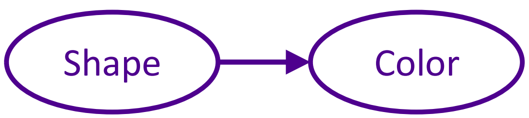
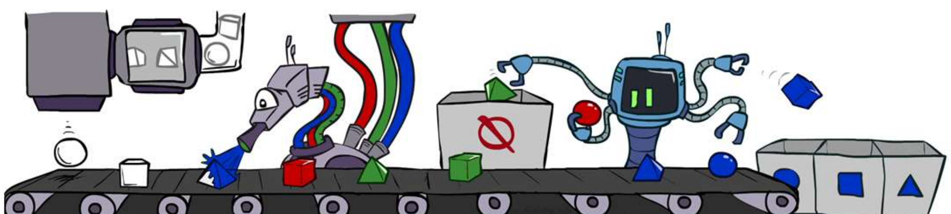
Likelihood Weighting

- Problem with rejection sampling:
  - If evidence is unlikely, rejects lots of samples
  - Evidence not exploited as you sample
  - Consider $P(\text{Shape}|\text{blue})$

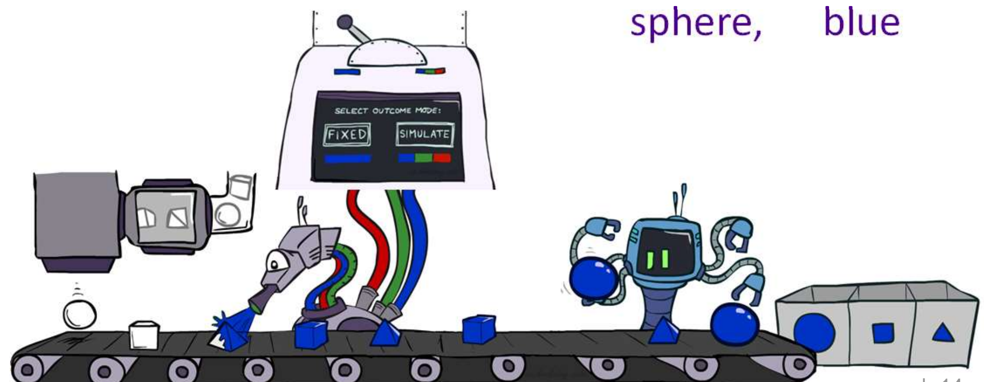


~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



- Idea: fix evidence variables and sample the rest
  - Problem: sample distribution not consistent!
  - Solution: weight by probability of evidence given parents

pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



Likelihood Weighting

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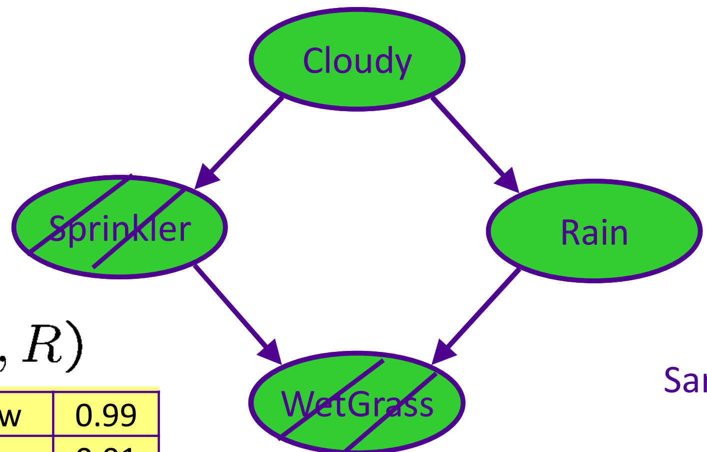

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-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

+c, +s, +r, +w

...

$$w = 1.0 \times 0.1 \times 0.99$$

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Likelihood Weighting

IN: evidence instantiation

$w = 1.0$

for $i = 1, 2, \dots, n$

    if X_i is an evidence variable

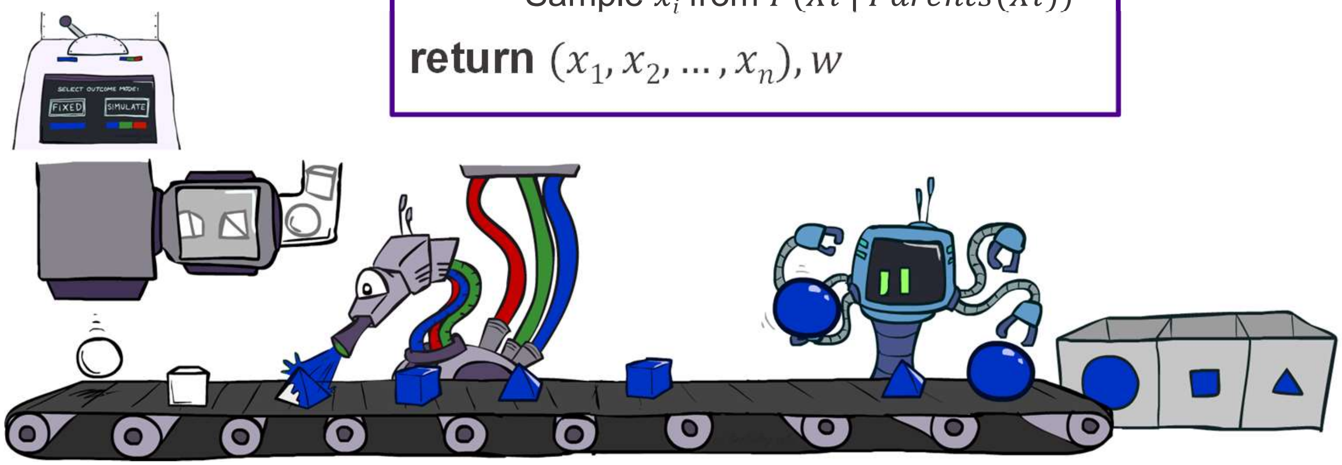
        X_i = observation x_i for X_i

        Set $w = w \times P(xi \mid Parents(Xi))$

    else

        Sample x_i from $P(Xi \mid Parents(Xi))$

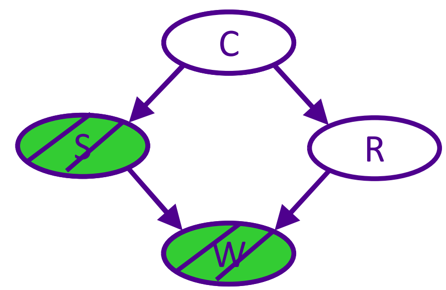
return $(x_1, x_2, \dots, x_n), w$



Likelihood Weighting

- Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i|\text{Parents}(Z_i))$$



- Now, samples have weights

$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i|\text{Parents}(E_i))$$

- Together, weighted sampling distribution is consistent

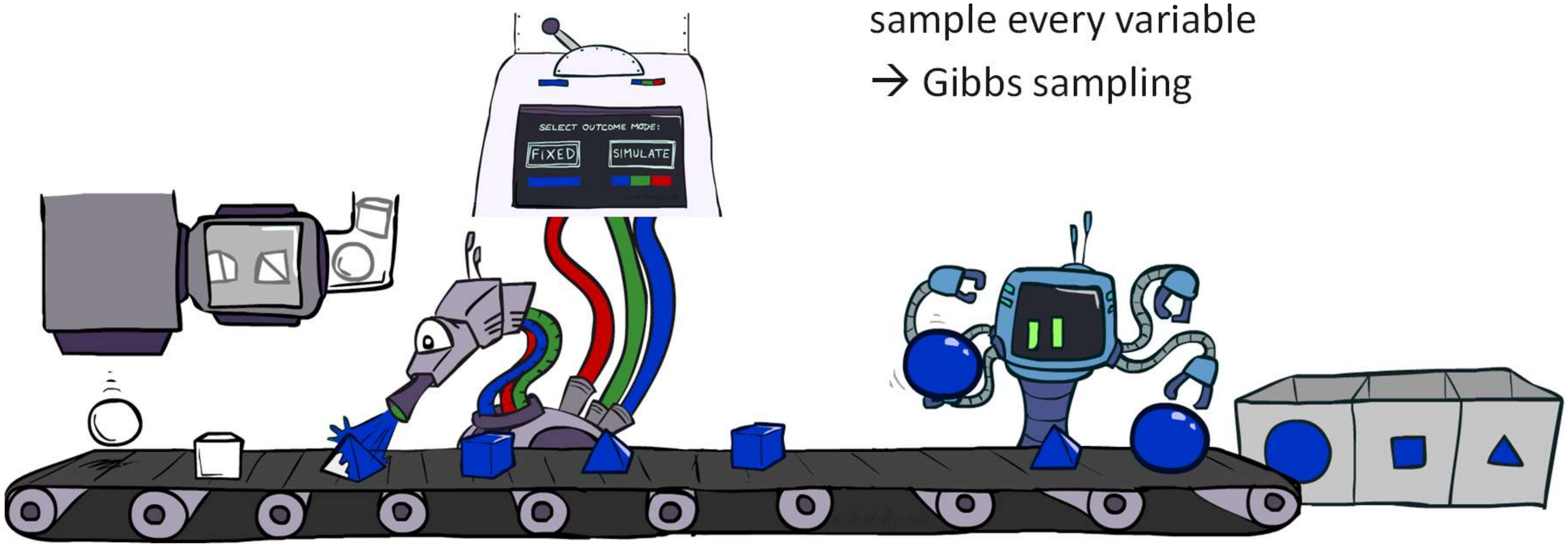
$$S_{\text{WS}}(z, e) \cdot w(z, e) = \prod_{i=1}^{l} P(z_i|\text{Parents}(z_i)) \prod_{i=1}^{m} P(e_i|\text{Parents}(e_i))$$
$$= P(\mathbf{z}, \mathbf{e})$$

Likelihood Weighting

- Likelihood weighting is good
  - We have taken evidence into account as we generate the sample
  - E.g. here, W's value will get picked based on the evidence values of S, R
  - More of our samples will reflect the state of the world suggested by the evidence

- Likelihood weighting doesn't solve all our problems
  - Evidence influences the choice of downstream variables, but not upstream ones (C isn't more likely to get a value matching the evidence)
- We would like to consider evidence when we sample every variable

  → Gibbs sampling



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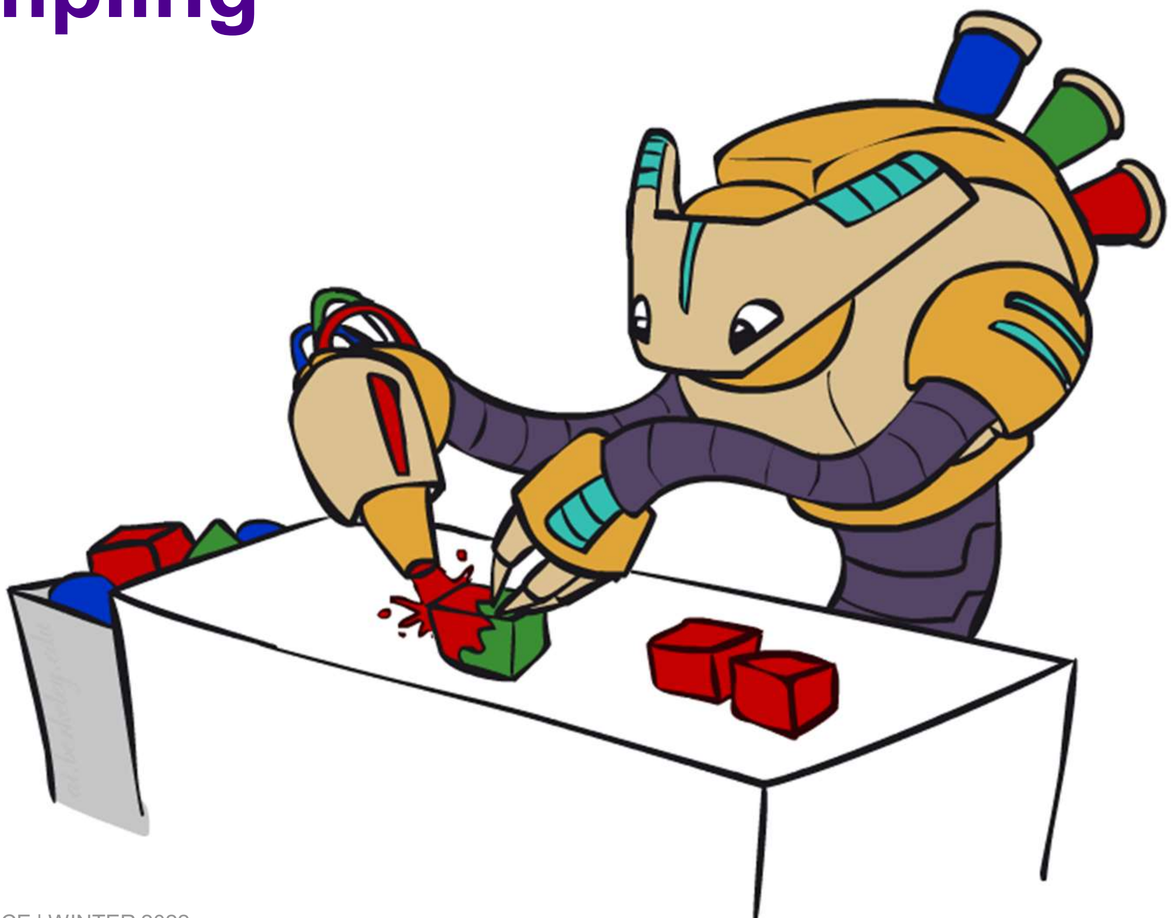
Gibbs Sampling



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Gibbs Sampling

- ***Procedure:***

1. keep track of a full instantiation $x_1, x_2, \dots, x_n$.
2. Start with an arbitrary instantiation consistent with the evidence.
3. Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
4. Keep repeating this for a long time.

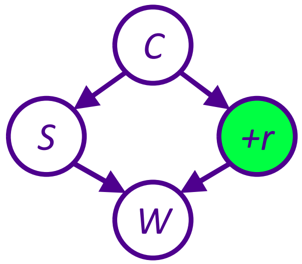
- ***Property:*** in the limit of repeating this infinitely many times the resulting sample is coming from the correct distribution

- ***Rationale*****:** both upstream and downstream variables condition on evidence.

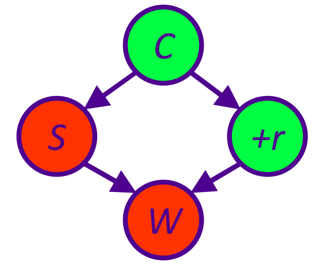
- **In contrast:**
  - likelihood weighting only conditions on upstream evidence, and hence weights obtained in likelihood weighting can sometimes be very small.
  - Sum of weights over all samples is indicative of how many “effective” samples were obtained, so want high weight.

Gibbs Sampling Example: $P(S|+r)$

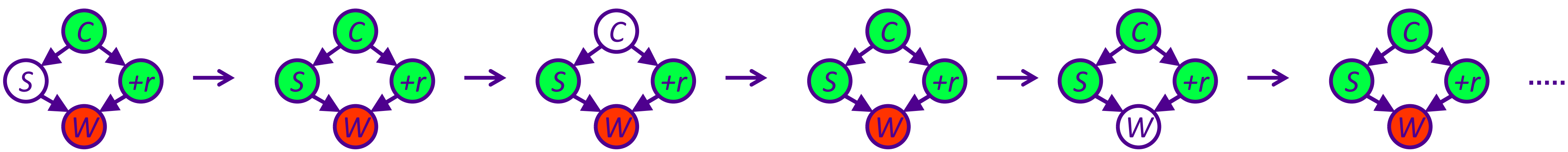
- Step 1: Fix evidence
  - $R = +r$



- Step 2: Initialize other variables
  - Randomly



- Steps 3: Repeat
  - Choose a non-evidence variable X
  - Resample X from $P(\ X$ | all other variables)



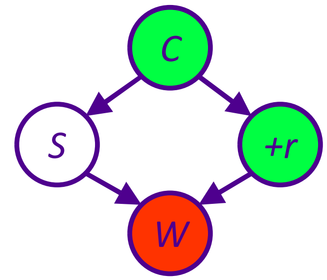
Sample from $P(S|+c,-w,+r)$ Sample from $P(C|+s,-w,+r)$ Sample from $P(W|+s,+c,+r)$

Gibbs Sampling

- How is this better than sampling from the full joint?
  - In a Bayes' Net, sampling a variable given all the other variables (e.g. $P(R|S,C,W)$) is usually much easier than sampling from the full joint distribution
    - Only requires a join on the variable to be sampled (in this case, a join on R)
    - The resulting factor only depends on the variable's parents, its children, and its children's parents (this is often referred to as its Markov blanket)

Efficient Resampling of One Variable

- Sample from $P(S \mid +c, +r, -w)$



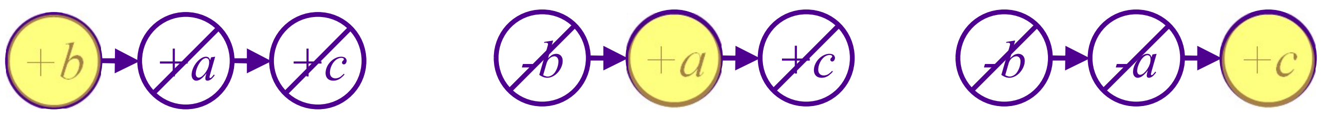
$$P(S|+c,+r,-w) = \frac{P(S,+c,+r,-w)}{P(+c,+r,-w)}$$
$$= \frac{P(S,+c,+r,-w)}{\sum_s P(s,+c,+r,-w)}$$
$$= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{\sum_s P(+c)P(s|+c)P(+r|+c)P(-w|s,+r)}$$
$$= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{P(+c)P(+r|+c)\sum_s P(s|+c)P(-w|s,+r)}$$
$$= \frac{P(S|+c)P(-w|S,+r)}{\sum_s P(s|+c)P(-w|s,+r)}$$

- Many things cancel out – only CPTs with S remain!
- More generally: only CPTs that have resampled variable need to be considered, and joined together

Markov Chain Monte Carlo*

- *Idea*: instead of sampling from scratch, create samples that are each like the last one.

- *Procedure*: resample one variable at a time, conditioned on all the rest, but keep evidence fixed. E.g., for $P(b|c)$:

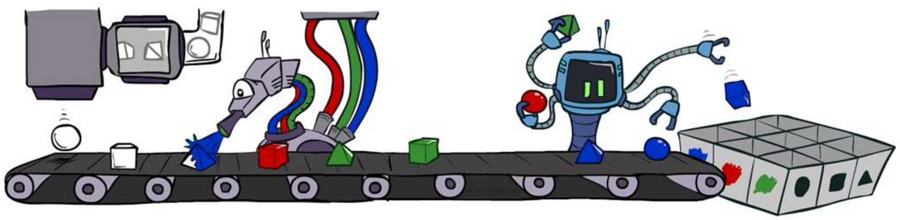


- *Properties*: Now samples are not independent (in fact they're nearly identical), but sample averages are still consistent estimators!

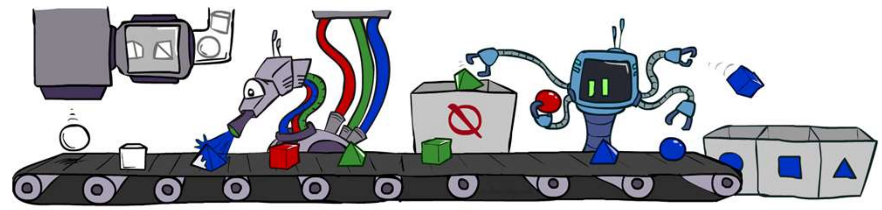
- *What's the point*: both upstream and downstream variables condition on evidence.

Bayes' Net Sampling Summary

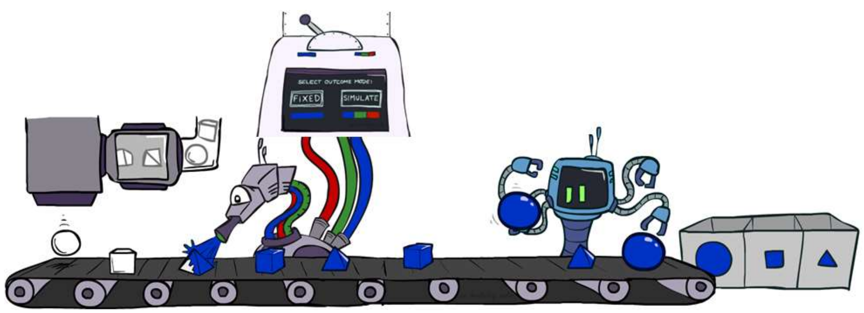
- Prior Sampling P



- Rejection Sampling $P(Q \mid e)$



- Likelihood Weighting $P(Q \mid e)$



- Gibbs Sampling $P(Q \mid e)$